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HEAT CONDUCTION WITH SOLIDIFICATION AND A CONVECTIVE BOUNDARY CONDITION AT THE FREEZING FRONT

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INTRODUCTION

THE PHENOMENON of heat transfer from a surface at a temperature below the fusion temperature of the surrounding medium has received much attention in the literature. However almost all analyses of such problems, involving the formation of a solid on a cold surface, have ignored the effects of convection.

Recently Libby and Chen [1] have presented an approximate solution which does take into consideration the effects of convective heating. Their analysis, based on Goodman's integral method [2], leads to a nonlinear second-order ordinary differential equation relating H, the dimensionless thickness of the deposited solid layer, to dimensionless time τ. This equation is then solved numerically for a particular set of the parameters that characterize the problem.

An alternate approximate technique which is applicable to this problem has been developed by Biot [3, 4]. Using Biot's technique a much simpler differential equation is obtained relating H and τ. This equation can be solved explicitly and for the particular case presented by Libby and Chen the two solutions are indistinguishable. Since the solution obtained following Biot's procedure gives a simple formula which is valid for all values of the parameters involved and since it eliminates the need for a computer, it is given here.

ANALYSIS

Consider a flat plate immersed in an infinite fluid initially at a uniform temperature. With respect to the coordinate system shown in Fig. 1, for $t > 0$ let $T(y, 0, t) = T_R$ = constant $< T_f$ = the fusion temperature of the medium. In this situation a solid is deposited on the plate and if the fluid flows over the plate in the positive y-direction due to either a forced external flow or natural convection (if the plate is vertical), the fluid–solid interface will be as shown in Fig. 1.

If the thickness of the solid is denoted by $h(y, t)$ then

$$T(y, h, t) = T_f$$

and due to the change of phase at the interface

$$k \frac{\partial T}{\partial z}(y, h, t) = q_c + \rho L\dot{h}$$

where q_c is the rate per unit area normal to the z-direction at which heat is transferred to the solid by convection.

Following Libby and Chen it is assumed that conduction in the solid may be treated as one-dimensional so that the energy equation becomes

$$\partial T/\partial t = \alpha\, \partial^2 T/\partial z^2 \qquad 0 < z < h$$

where it has been assumed that α, the thermal diffusivity of the deposit, is constant. It is also argued in reference [1] that

$$q_c = q_{c,0} + \pi\dot{h}$$

where π is a constant and $q_{c,0} = q_{c,0}(y)$.

Thus the problem as formulated by Libby and Chen leads to the mathematical system:

$$\partial T/\partial t = \alpha\, \partial^2 T/\partial z^2 \qquad 0 < z < h \tag{1}$$

$$T(y, 0, t) = T_R \tag{1a}$$

$$T(y, h, t) = T_f \tag{1b}$$

$$k \frac{\partial T}{\partial z}(y, h, t) = q_{c,0} + (\pi + \rho L)\,\dot{h} \tag{1c}$$

Following Biot, Q, a heat flux variable, is defined by

$$Q = \int_0^t - k \frac{\partial T}{\partial z}\, dt \tag{2}$$

so that

$$\partial Q/\partial t = -k\, \partial T/\partial z \tag{2a}$$

$$\partial Q/\partial z = -\rho c(T - T_0) = -\rho c[T(y, z, t) - T(y, z, 0)] \tag{2b}$$

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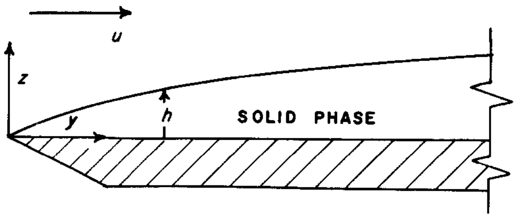

FIG. 1. Schematic representation of the coordinate system and the frozen deposit.

If Q is written as a function $Q(q_1, \ldots, q_n; y, z, t)$ of generalized coordinates $q_1, \ldots, q_n$, it is easily verified that for this particular case Biot's variational equations can be written in the form

$$\int_0^h \left[\frac{1}{\alpha}\frac{\partial Q}{\partial t}\frac{\partial Q}{\partial q_k} + \frac{\partial Q}{\partial z}\frac{\partial}{\partial q_k}\left(\frac{\partial Q}{\partial z}\right)\right] \mathrm{d}z = \frac{\partial Q}{\partial z}\frac{\partial Q}{\partial q_k}\Bigg|_0^h \qquad k = 1, \ldots, n \quad (3)$$

In terms of Q the boundary conditions are

$$\frac{\partial Q}{\partial z}(y, 0, t) = -\rho c(T_R - T_f) \quad (3a)$$

$$\frac{\partial Q}{\partial z}(y, h, t) = 0 \quad (3b)$$

$$\frac{\partial Q}{\partial t}(y, h, t) = -q_{c,0} - (\pi + \rho L)\,\dot{h} \quad (3c)$$

where the initial temperature of the solid, T_0, has been taken as T_f.

Defining

$$\xi = \frac{z}{h(y, t)}, \qquad \theta = T_f/T_R, \qquad \bar{L} = \alpha(\pi + \rho L)/kT_f$$

and satisfying the boundary conditions for Q with a second-order polynomial in ξ

$$Q = q_0(y, t) + q_1(y, t)\,\xi + q_2(y, t)\,\xi^2$$

it is found that

$$Q = -q_{c,0}t - \frac{kT_R h}{\alpha}\left[\theta\bar{L} + \tfrac{1}{2}(\theta - 1)(\xi - 1)^2\right] \quad (4)$$

in which $h(y, t)$ is the only unknown generalized coordinate.

Substituting equation (4) into (3) and defining

$$H = \frac{hq_{c,0}}{kT_R}, \qquad \tau = \alpha t(q_{c,0}/kT_R)^2, \qquad \eta = \frac{\theta - 1}{\theta\bar{L}}$$

the following equation is obtained for H:

$$AH\frac{\mathrm{d}H}{\mathrm{d}\tau} + BH = C \quad (5)$$

in which

$$A = 2\eta^2 + 10\eta + 15, \qquad B = 5(\eta + 3)/\theta\bar{L}, \qquad C = 5\eta(\eta + 3).$$

Equation (5) can be integrated to give

$$\tau = (1 - \theta)f(\eta)\left[H + (\theta - 1)\ln\left(1 - \frac{H}{\theta - 1}\right)\right] \quad (6)$$

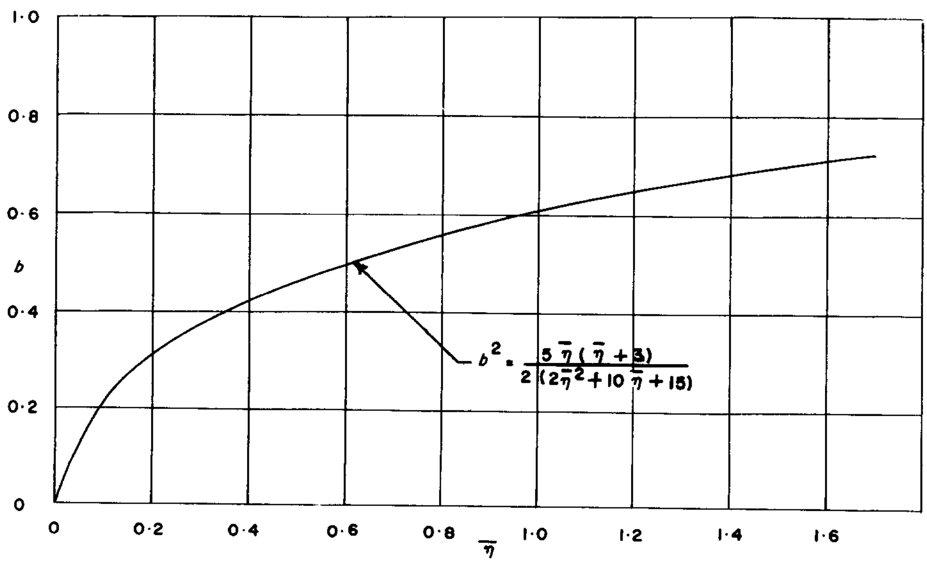

FIG. 2. Approximate solution for small time.

where

$$f(\eta) = \frac{2\eta^2 + 10\eta - 15}{5\eta(\eta + 3)}$$

It is easily shown that equation (6) and the previous solution [1] both are in agreement with exact solutions for the limiting cases of $\tau \to 0$ and $\tau \to \infty$. Moreover for the particular example presented in reference [1], the results of the two approximate solutions are indistinguishable over the entire range of τ

For $\tau \to 0$ it would be expected on physical grounds that convective currents could be ignored. In this case the exact solution [5] is given by

$$H = 2b\sqrt{\tau}$$

where the constant b must be found from the transcendental equation

$$b \exp[b^2] \operatorname{erf} b = 1{\cdot}77\bar{\eta}; \qquad \eta = \eta_{\pi=0}$$

For $\tau \to 0$, equation (6) can be written as

$$H = \sqrt{[2\tau \cdot f(\eta)]} \qquad (6a)$$

Thus for $\tau \to 0$, equation (6a) gives the approximation

$$b^2 = \tfrac{1}{2} f(\bar{\eta}) \qquad (7)$$

where π is set equal to zero since convection is ignored in the classical solution.

For $\tau \to \infty$, $\partial h/\partial t$, $\partial T/\partial t \to 0$ so that from equations (1) and (1c) it is easily verified that for this limiting case $H \to \theta - 1$. Equations (7) and (6) are plotted in Figs. 2 and 3

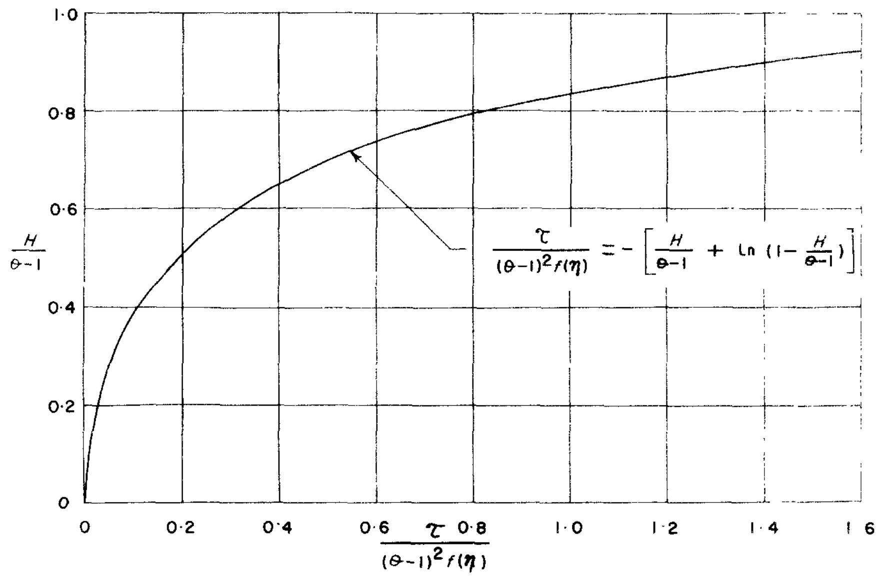

FIG. 3. Thickness of the deposit at a fixed position along the flat plate.

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